1. 留数

$$\mathrm{Res}(f,a) = \frac{1}{2\pi i}\int_{|z-a|=\rho} f(z)\,dz$$

若 $f(z) = \sum_{-\infty}^{\infty} c_n (z-a)^n$, $\mathrm{Res}(f,a) = c_{-1}$

若 $f(z) = \frac{1}{(z-a)^m}\, g(z)$, $g(z)$ 在 a 附近全纯.

则 $\mathrm{Res}(f,a) = \frac{1}{(m-1)!}\, g^{(m-1)}(a)$

留数定理.

f 在 γ 内部 $D\setminus\{z_1, \cdots z_k\}$ 全纯, 在 $\overline{D}\setminus\{z_1 \cdots z_k\}$ 连续.

则 $\int_\gamma f\,dz = 2\pi i \sum_{k=1}^{n} \mathrm{Res}(f, z_k)$.

∞ 点留数.

f 在 $|z|>R$ 全纯.

$$\mathrm{Res}(f,\infty) = \underset{\Delta}{-}\frac{1}{2\pi i}\int_{|z|=\rho} f(z)\,dz, \quad \rho > R$$

若 $f=\sum_{-\infty}^{\infty} c_n z^n$, 则 $\mathrm{Res}(f,\infty) = -c_{-1}$

Theorem

$f \in H(\mathbb{C}\setminus\{z_1 \cdots z_n\})$, 则

$$\sum_{k=1}^{n} \mathrm{Res}(f, z_k) + \mathrm{Res}(f,\infty) = 0$$

引理.

$\lim_{z\to\infty} f(z) = 0$. 则 $\lim_{R\to\infty}\int_{\gamma_R} e^{i\alpha z} f(z) = 0$, $\forall \alpha > 0$

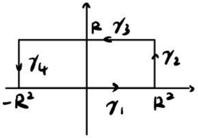
且 $\int_{-\infty}^{\infty} f(x)\, e^{i\alpha x}\, dx = 2\pi i \sum_k \mathrm{Res}\,(e^{i\alpha z} f(z), a_k)$.

计算.

(1) $I=\int_{-\infty}^{\infty}\frac{\cos ax}{b^2+x^2}\,dx$, $a,b>0$

$f(z)=\frac{e^{iaz}}{b^2+z^2}$, $I=\int_{\mathbb{R}} f(z)\,dz$

$z=x+iy$, $f(x+iy)=\frac{e^{-ay}\cdot a^{iax}}{b^2+x^2-y^2+2ixy}$



$\int_{\gamma_2}|f(z)|dz \le \int_0^R \frac{1}{b^2+R^4-R^2}\,dy \to 0$

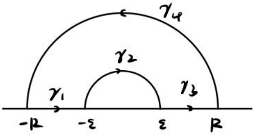
as $R\to\infty$

同理, $\int_{\gamma_4}|f(z)|dz \to 0$

$$\left|\int_{\gamma_3} f(z)\,dz\right| \le \int_{-R^2}^{R^2}\frac{e^{-aR}}{x^2-R^2}\,dx = O(e^{-aR}) \to 0$$

故 $\int_{\gamma_4} f(z)\,dz \to 2\pi i\,\mathrm{Res}(f,bi)=\frac{\pi}{b}e^{-ab}$

(2) $I=\int_{-\infty}^{\infty}\frac{\sin x}{x}\,dx$



$f(z)=\frac{e^{iz}}{z}$, $I=\mathrm{Im}\left(\int_{\mathbb{R}} f\,dz\right)$

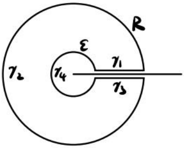
引理 $\int_{\gamma_4} f \to 0$

$\int_0^{\pi}\frac{e^{i\rho e^{i\theta}}}{\rho e^{i\theta}}\,\rho i e^{i\theta}\,d\theta=\int_0^{\pi}e^{i\rho e^{i\theta}}\cdot i$

$\to \pi i$

故 $I=\pi$.

(3) $I=\int_0^{\infty}\frac{dx}{(1+x)\,x^{\alpha}}$, $0<\alpha<1$



$F(z)=\frac{1}{(1+z)\,z^{\alpha}}$

$\int_{\gamma} F(z)\,dz = 2\pi i\,\mathrm{Res}(F(z),-1)$

$$z^{\alpha}\Big|_{z=x_{\text{下}}} = |x|^{\alpha}e^{i\alpha\,\mathrm{Arg}\,x_{\text{下}}} = |x|^{\alpha}e^{i\alpha(\mathrm{Arg}\,x_{\text{上}}+\Delta_C \mathrm{Arg}\,z)}$$

$$=|x|^{\alpha}\cdot e^{2\pi\alpha\cdot i}$$

$$\Rightarrow \int_{\gamma_3} F dz = - e^{-2\pi a i} \int_{\gamma_1} F dz.$$

易知 $\left|\int_{\gamma_2} F\, dz\right| \to 0$，$\left|\int_{\gamma_4} F\, dz\right| \to 0$

$$\mathrm{Res}(F(z), -1) = (1+z)F(z)\Big|_{z=-1} = z^{-\alpha}\Big|_{z=-1}$$

$$= e^{-\alpha i \,\mathrm{Arg}(-1)} = e^{-\alpha(0+\pi)i} = e^{-\pi\alpha i}$$

故 $\displaystyle\int_0^\infty F(x)dx = \frac{2\pi i e^{-\pi\alpha i}}{1-e^{-2\pi a i}} = \frac{\pi}{\sin\pi a}$

2. 全纯开拓.

Painlevé 原理.

$$H(D \setminus \bigcup_{k=1}^{n} \gamma_k) \cap C(D) \Rightarrow H(D)$$

Schwarz 对称

D关于实轴对称，$f \in H(D\cap \overset{\text{上半平面}}{\mathbb{H}}) \cap C(\bar{D}\cap\bar{\mathbb{H}})$

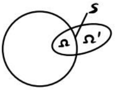
f在 $D\cap\mathbb{R}$ 上实值，则

$$F(z) = \begin{cases} f(z) \\ \overline{f(z)} \end{cases} \quad \text{在}D\text{全纯}$$

$$\boxed{\text{对称点}\ z^* = a + \frac{R^2}{\bar{z}-\bar{a}}}$$

定理.

Ω，Ω' 关于 $|z|=r$ 对称，f在Ω全纯，$f(s) \subset B(b,R)$



$$F(z) = \begin{cases} f(z), & z\in\Omega \\ b + \dfrac{R^2}{\overline{f(a+\frac{r^2}{\bar{z}-\bar{a}})} - \bar{b}}, & z\in\Omega' \end{cases}$$

幂级数 正则点
奇异点

1、幂级数 收敛圆周至少一个奇异点

2. 若 $\lim_{z\to z_0}|f(z)|=\infty$，则 z_0 奇异.

3. $f=\sum_{n=0}^{\infty}c_nz^n$，收敛半径 R，$|z_0|=R$

若 z_0 正则，取 z_0' 在 0 z_0 连线上，$f(z)=\sum_{n=0}^{\infty}c_n'(z-z_0')^n$

收敛半径 ρ.

(1) $\rho>R-|z_0'|\Rightarrow z_0$ 正则

(2) $\rho=R-|z_0'|\Rightarrow z_0$ 奇异.

3. Riemann 映照.

Hurwitz

若 $f_n\in H(D)$，$f_n\to f$ 内闭一致，f 不恒为 0，γ 闭，不过 f 零点，$\exists N$，$\forall n\geq N$，f_n 与 f 在 γ 内零点数相同.

Arzela – Ascoli

$K\subset\subset\mathbb{C}$ compact. f_n 在 K 一致有界，等度连续

则 $\exists k_n$，$\{f_{k_n}\}$ 一致收敛.

Montel.

$f_n\in H(D)$，且在 D 内闭一致有界，则 $\exists k_n$，

$\{f_{k_n}\}$ 内闭一致收敛.

Riemann.

$G\subsetneq\mathbb{C}$ 单连通域，$\forall z_0\in G$，$\exists!\ f:G\to\mathbb{D}$, s.t.

(1) f 在 G 中全纯，单叶

(2) $f(z_0)=0$

(3) $f'(z_0)>0$

Proof of Riemann theorem

1、取 $a\in\mathbb{C}\setminus G$. $g(z)=\log(z-a)$ 在 G 上有单值分支.

(1) g 在 G 上全纯且单叶

(2) 令 $w_0=g(z_0)$，则 $\tilde{w}_0=w_0+2\pi i\notin g(G)$

(3) $\exists\delta>0$，$B(\tilde{w}_0,\delta)\cap g(G)=\phi$

否则可得一列 $z_n \to z_0$，$g(z_n) \to \widehat{w_0}$，与 $g(z_n) \to g(z_0)$ 矛盾.

(4) 令 $f(z) = \frac{1}{g(z) - \widehat{w_0}} + \frac{1}{2\pi i}$，则 $f(z_0) = 0$，f 有界，$|f| < C$.

$h = \frac{1}{C} f$，则 $h(z_0) = 0$，$|h(z)| < 1$，$\forall z \in G$.

以上转化为 $G \subset \mathbb{D}$ 情形.

2. 考虑 $0 \in G \subset \mathbb{D}$ 情形. 考虑 $\mathcal{F} = \{$单叶全纯 $f: \Omega \to \mathbb{D} \mid f(0) = 0\}$

则 f 非空 (Id) 且一致有界

(1) $\{|f'(0)| \mid f \in \mathcal{F}\}$ 一致有界.

$B(0,\delta) \subset \Omega$，$|f'(0)| \leq \frac{1}{2\pi} \left| \int_{\partial B(0,\delta)} \frac{f(z)}{z} dz \right| \leq \frac{1}{\delta}$

(2) $\lambda = \sup\{|f'(0)| \mid f \in \mathcal{F}\}$，$\exists \{f_n\} \in \mathcal{F}$，$|f_n'(0)| \to \lambda$

Montel，$\exists f_{n_k} \to f_\infty$，内闭一致.

则 (a) $|f_\infty'(0)| = \lambda$，$f_\infty(0) = 0$

(b) $|f_\infty(z)| < 1$，$\forall z \in G$.

(c) 由 Hurwitz 定理，f_∞ 单叶

(3) $f_\infty(G) = \mathbb{D}$.

否则，$\alpha \in \mathbb{D} \setminus f_\infty(G)$，$\varphi_\alpha(z) = \frac{\alpha - z}{1 - \bar{\alpha} z}$，$\mathbb{D} \to \mathbb{D}$，$\alpha \to 0$

$\varphi_\alpha \circ f_\infty$ 是全纯单叶函数，且 $\varphi_\alpha \circ f_\infty(G)$ 单连通，

$0 \notin \varphi_\alpha \circ f_\infty(G)$.

故定义 $R(z) = \sqrt{\varphi_\alpha \circ f_\infty(z)}$，则 R 在 G 上有单值分支，

且单叶.

令 $\beta = \sqrt{\alpha}$，$\varphi_\beta(z) = \frac{\beta - z}{1 - \bar{\beta} z}$，$F(z) = \varphi_\beta \circ R(z)$

则 $F: G \to \mathbb{D}$ 单叶全纯，$F(0) = 0$.

$f_\infty(z) = \varphi_\alpha^{-1} \circ S^{-1} \circ \varphi_\beta^{-1} \circ F(z) = T \circ F(z)$ $(S(w) = \sqrt{w})$

$S^{-1}(z) = z^2: \mathbb{D} \to \mathbb{D}$ 全纯，故 $T: \mathbb{D} \to \mathbb{D}$ 全纯，$T(0) = 0$

且非单叶，故 $|T'(0)| < 1$

$\lambda = |f_\infty'(0)| = |T'(0)| \cdot |F'(0)| < |F'(0)|$，矛盾.

故 $\exists f: G \to \mathbb{D}$ 双全纯. $f \cdot e^{-i\operatorname{Arg} f'(z_0)}$ 为所求.

3、 $G \subsetneq \mathbb{C}$，若 $f_1, f_2: G \to \mathbb{D}$ 为符合要求的全纯同构.

$T = f_2 \circ f_1^{-1}: \mathbb{D} \to \mathbb{D}$，$0 \mapsto 0$ $\Rightarrow$ $T(z) = e^{i\theta}$ $\Rightarrow f_1 = f_2$

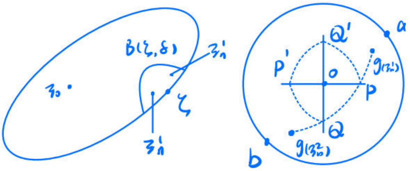
边界对应定理.

G 是一条简单闭曲线 Γ 所围成的域，若 $w = f(z)$ 将 G 双全纯映为 $B(0,1)$，则 f 可扩充到 Γ 上，s.t. $f \in C(\overline{G})$，且把 Γ 一一地映为 $|w|=1$，Γ 关于 G 的正向对应于 $f(\Gamma)$ 关于 $B(0,1)$ 的正向.

1. $\forall \zeta \in \partial G$，$\lim\limits_{\substack{z \to \zeta \\ z \in G}} f(z)$ 存在

即 $z_n^1 \to \zeta$，$z_n^2 \to \zeta$，$f(z_n^1) \to a$，$f(z_n^2) \to b \Rightarrow a = b$

显然地，$a, b \in \partial B(0,1)$，取分式线性变换 $\varphi: a \to e^{\frac{\pi}{4} i}$，



$b \to e^{\frac{5}{4}\pi i}$

$B(0,1) \to B(0,1)$

$g = \varphi \circ f: G \to B(0,1)$

$z_0 = g^{-1}(0)$

$\forall \varepsilon > 0$，$\exists \delta > 0$，$\delta < d(\zeta, z_0)$，$\exists z_n^1, z_n^2 \in B(\zeta, \delta) \cap G$

$g(z_n^1) \in B(a, \varepsilon)$，$g(z_n^2) \in B(b, \varepsilon)$

取 l connect z_n^1, z_n^2，$L = g(l)$ intersect

$\mathbb{R}$，$i\mathbb{R}$ 于 P，Q，对称后得 P'，Q'，四段弧围成区域 D，

$0 \in D$.

$$F(w) = (g^{-1}(w) - \zeta)(\overline{g^{-1}(\bar{w})} - \bar{\zeta})\cdot$$
$$(g^{-1}(-w) - \zeta)(\overline{g^{-1}(-\bar{w})} - \bar{\zeta})$$

$|g^{-1}(w) - \zeta| < \delta$，记 $M = \mathrm{diam}(G)$

$\Rightarrow |F(w)| < \delta \cdot M^3 \Rightarrow |F(0)| < \delta \cdot M^3$，对 δ 充分小

$\Rightarrow F(0) = 0 \Rightarrow g^{-1}(0) = \zeta$，矛盾

故 $a = b$

2. 定义 $f(\zeta) = \lim\limits_{\substack{z \to \zeta \\ z \in G}} f(z)$，$\forall \zeta \in \partial G$

则 $f \in C(\bar{G})$，只需 $\lim\limits_{\substack{z \to \zeta \\ z \in \partial G}} f(z) = f(\zeta)$，易证.

3. $\zeta, \zeta' \in \partial G$，$\zeta \neq \zeta' \Rightarrow f(\zeta) \neq f(\zeta')$

4. 保角性 $\Rightarrow$ 保持边界方向

定理

G, D 分别为可求长简单闭曲线 γ, Γ 围成，$f \in H(G) \cap C(\bar{G})$，将 γ 一一映为 Γ，则 $w = f(z)$ 将 G 一一映为 D，γ 关于 G 正向对应 Γ 关于 D 正向.

$\forall w_0 \in D$，$f(z) - w_0$ 在 G 中零点数

$N = \frac{1}{2\pi} \Delta_\gamma \mathrm{Arg}(f(z) - w_0) = \frac{1}{2\pi} \Delta_\Gamma \mathrm{Arg}(w - w_0) = \pm 1$

$\Rightarrow N = 1 \Rightarrow f(z) - w_0$ 在 G 中恰有一零点

$\forall w_0 \notin \bar{D}$，不存在 $z \in G$，$f(z) = w_0$

4. Fourier 变换.

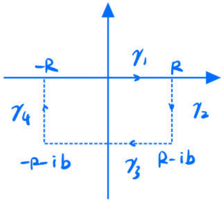
$$\hat{f}(\xi) = \int_{-\infty}^{\infty} f(x)\cdot e^{-2\pi i x \xi}\,dx\,,\quad f(x) = \int_{-\infty}^{\infty} \hat{f}(\xi)\, e^{2\pi i x \xi}\, d\xi$$

$\forall a > 0,\quad S_a = \{ z \in \mathbb{C} \mid |\mathrm{Im}(z)| < a \}$

$\mathcal{F}_a = \{ f \in H(S_a) \mid \exists A > 0, \forall x \in \mathbb{R}, |y| < a,\ |f(x+iy)| \le \frac{A}{1+x^2} \}$

$\mathcal{F} = \bigcup_{a<\infty} \mathcal{F}_a$

(1) $f \in \mathcal{F}_a$, $\forall\ b \in [0, a]$, $|\hat{f}(\xi)| \le B \cdot e^{-2\pi b |\xi|}$, $\xi \in \mathbb{R}$.



$$\left| \int_{\gamma_2} f(z)\, e^{-2\pi i z \xi}\, dz \right|$$

$$\le \int_0^b \left| f(-R-it)\, e^{-2\pi i(-R-it)\xi} \right| dt$$

$$\le \int_0^b \frac{A}{1+R^2}\, e^{-2\pi t \xi}\, dt \longrightarrow 0$$

同理, $\left| \int_{\gamma_4} f(z)\, e^{-2\pi i z \xi}\, dz \right| \longrightarrow 0$

故 $|f(\xi)| = \left| \int_{-\infty}^{\infty} f(x-ib)\, e^{-2\pi i (x-ib)\xi}\, dx \right| \le \int_{-\infty}^{\infty} \frac{A}{1+x^2}\, e^{-2\pi b \xi}\, dx$

$\le B \cdot e^{-2\pi b \xi}$

(2) $f \in \mathcal{F}$, 则 $\forall x \in \mathbb{R}$, $f(x) = \int_{-\infty}^{\infty} \hat{f}(\xi)\, e^{2\pi i x \xi}\, d\xi$

lemma.

$A > 0$, $B \in \mathbb{R}$, $\int_0^{\infty} e^{-(A+iB)\xi}\, d\xi = \frac{1}{A+iB}$

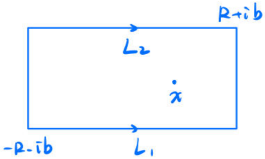
$$\int_{-\infty}^{\infty} \hat{f}(\xi)\, e^{2\pi i x \xi}\, d\xi = \int_{-\infty}^{0} \hat{f}(\xi)\, e^{2\pi i x \xi}\, d\xi + \int_0^{\infty} \hat{f}(\xi)\, e^{2\pi i x \xi}\, d\xi$$

$$\int_0^\infty \hat f(\xi)\, e^{2\pi i x\xi}\, d\xi = \int_0^\infty \int_{-\infty}^\infty f(u-ib)\, e^{-2\pi i(u-ib)\xi}\, e^{2\pi i x\xi}\, du\, d\xi$$

$$= \int_{-\infty}^\infty f(u-ib)\, \frac{1}{2\pi b + 2\pi i(u-x)}\, du$$

$$= \frac{1}{2\pi i}\int_{-\infty}^\infty \frac{f(u-ib)}{u-ib-x}\, du = \frac{1}{2\pi i}\int_{L_1} \frac{f(\zeta)}{\zeta - x}\, d\zeta,$$

同理，$\int_{-\infty}^0 \hat f(\xi)\, e^{2\pi i x\xi}\, d\xi = -\frac{1}{2\pi i}\int_{L_2}\frac{f(\zeta)}{\zeta-x}\, d\zeta$



$$f(x) = \frac{1}{2\pi i}\int_{L_1}\frac{f(\zeta)}{\zeta-x}\, d\zeta - \frac{1}{2\pi i}\int_{L_2}\frac{f(\zeta)}{\zeta-x}\, d\zeta$$

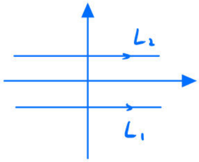
即证.

(3) Poisson 公式

$$f \in \mathcal{F} \Rightarrow \sum_{n\in\mathbb{Z}} f(n) = \sum_{n\in\mathbb{Z}} \hat f(n)$$

$$\sum_{|n|\le N} f(n) = \int_{\gamma_N} \frac{f(z)}{e^{2\pi i z}-1}\, dz$$

$$\sum_{n\in\mathbb{Z}} f(n) = \int_{L_1}\frac{f(z)}{e^{2\pi i z}-1}\, dz - \int_{L_2}\frac{f(z)}{e^{2\pi i z}-1}\, dz$$



$$= \int_{L_1} f(z)\left(e^{-2\pi i z}\sum_{n=0}^\infty e^{-2\pi i n z}\right) dz$$

$$+ \int_{L_2} f(z)\left(\sum_{n=0}^\infty e^{2\pi i n z}\right) dz$$

$$= \sum_{n\in\mathbb{Z}} \int_{-\infty}^\infty f(x)\, e^{-2\pi i n z}\, dz = \sum_{n\in\mathbb{Z}} \hat f(n)$$

(4) Paley - Wiener 定理.

(a) $\hat f$ 满足 $|\hat f(\xi)| \le A e^{-2\pi a|\xi|}$, $a, A > 0$, 则 $\forall\, b \in (0, a)$,

$$f(z) = \int_{-\infty}^\infty \hat f(\xi)\, e^{2\pi i \xi z}\, d\xi \in H(S_b)$$

$$f_n(z) = \int_{-n}^n \hat f(\xi)\, e^{2\pi i \xi z}\, d\xi$$

$|f(z)-f_n(z)| \le A \int_{|\xi| \ge n} e^{-2\pi a|\xi|} e^{2\pi b|\xi|} d\xi \to 0$

$|\hat{f}(\xi) e^{2\pi i \xi z}| \le e^{-2\pi a|\xi|} e^{2\pi b|\xi|} \in \mathcal{L}^1(\mathbb{R}) \Rightarrow f$ 收敛

$f_n \rightrightarrows f \Rightarrow f$ 全纯

(b) f cts，在 $\mathbb{R}$ 缓减，则

f 扩充到复平面是整函数，且满足 $|f(z)| \le A e^{2\pi M|z|}$，

$$\Leftrightarrow \quad \operatorname{supp} \hat{f} \subset [-M, M]$$

若 $\operatorname{supp} \hat{f} \subset [-M, M]$，$g(z) = \int_{-M}^{M} \hat{f}(\xi) e^{2\pi i \xi z} d\xi$

$g|_{\mathbb{R}} = f$，$|g(z)| \le \int_{-M}^{M} |\hat{f}(\xi)| e^{-2\pi \xi y} d\xi \le A e^{2\pi M|z|}$

若 $|f(z)| \le A e^{2\pi M|z|}$

(1) 若 $f \in H(\mathbb{C})$，且 $|f(x+iy)| \le A' \dfrac{e^{2\pi M|y|}}{1+x^2}$

设 $\xi > M$，$\hat{f}(\xi) = \int_{-\infty}^{\infty} f(x) e^{-2\pi i \xi x}$

$= \int_{-\infty}^{\infty} f(x-iy) e^{-2\pi i \xi(x-iy)} dx$

$\Rightarrow |\hat{f}(\xi)| \le A' \int_{-\infty}^{\infty} \dfrac{e^{2\pi M y} - e^{2\pi \xi y}}{1+x^2} dx \le C \cdot e^{-2\pi y(\xi - M)}$

$y \to +\infty \Rightarrow \hat{f}(\xi) = 0$

(2) 若 $f \in H(\mathbb{C})$，$|f(x+iy)| \le A e^{2\pi M|y|}$

设 $\xi > M$，对 $\varepsilon > 0$，$f_\varepsilon(z) = \dfrac{f(z)}{(1-i\varepsilon z)^2}$

$|\hat{f}_\varepsilon(\xi) - \hat{f}(\xi)| \le \int_{-\infty}^{\infty} |f(x)| \cdot \left(\dfrac{1}{(1-i\varepsilon x)^2} - 1\right) dx$

f 缓减，故 $\hat{f}_\varepsilon(\xi) \to \hat{f}(\xi)$ as $\varepsilon \to 0$，

而 $\forall \varepsilon$，$|f_\varepsilon(x+iy)| \le A'' \dfrac{e^{2\pi M|y|}}{1+x^2} \Rightarrow \hat{f}_\varepsilon(\xi) = 0$，$\forall \xi > M$

$\Rightarrow \hat{f}(\xi) = 0$.

(3) Want to prove (2) 中条件成立

Fact. $\forall x \in \mathbb{R}$，$|f(x)| \le A$，$\forall z \in \mathbb{C}$，$|f(z)| \le B e^{2\pi M|z|}$

$\Rightarrow |f(x+iy)| \le C e^{2\pi M|y|}$

(c) F 在 $S = \{z : -\pi/4 < \arg z < \pi/4\}$ 全纯，在 $\overline{S}$ 上 cts，

在 ∂S 上，$|F| \le 1$ ，$\exists\, C, c > 0$，$|F(z)| \le Ce^{c|z|}$

$\Rightarrow \forall\, z \in S$，$|F(z)| \le 1$

令 $F_\varepsilon(z) = F(z) \cdot e^{-\varepsilon z^{3/2}} \in H(S) \cap C(\bar{S})$，

最大模原理 $|F_\varepsilon(z)| \le 1 \Rightarrow |F(z)| \le 1 \quad (\varepsilon \to 0)$.

(d) f, $\hat{f}$ 微减，

$\forall\, \xi < 0$，$\hat{f}(\xi) = 0 \quad \Leftrightarrow$

f 可延拓为 $\overline{H} = \{z = x + iy \mid y \ge 0\}$ 的连续有界函数
且在内部全纯 .

"$\Rightarrow$" $f(x) = \int_0^{+\infty} \hat{f}(\xi)\, e^{2\pi i x\xi}\, d\xi$

$f(z) = \int_0^{\infty} \hat{f}(\xi)\, e^{2\pi i z\xi}\, d\xi$，易知收敛，且由 $\int_0^{n} \hat{f}(\xi)\, e^{2\pi i z\xi}\, d\xi$
一致收敛 $\Rightarrow$ 可延拓 .

"$\Leftarrow$" $f_{\varepsilon,\delta}(z) = \dfrac{f(z + i\delta)}{(1 - i\varepsilon z)^2} \in H(\mathbb{H})$

$\xi < 0$，$\hat{f}_{\varepsilon,\delta}(\xi) = \int_{-\infty}^{\infty} \dfrac{f(x + i\delta)}{(1 - i\varepsilon x)^2}\, e^{-2\pi i x\xi}\, dx$

$$= \int_{-\infty}^{\infty} \frac{f(x + i\delta + iy)}{(1 + \varepsilon y - i\varepsilon x)^2}\, e^{-2\pi i (x + iy)\xi}\, dx$$

$$|\hat{f}_{\varepsilon,\delta}(\xi)| \le \int_{-\infty}^{\infty} \frac{M}{(1 + \varepsilon y)^2 + \varepsilon^2 x^2}\, e^{2\pi y\xi}\, e^{-2\pi i x\xi}\, dx$$

$\to 0$ as $y \to \infty$

$\Rightarrow \hat{f}_{\varepsilon,\delta}(\xi) = 0 \Rightarrow \hat{f}_{\varepsilon,0}(\xi) = 0 \Rightarrow$

$\hat{f}(\xi) = \hat{f}_{0,0}(\xi) = 0$，$\forall\, \xi < 0$

5. 整函数

(1) 圆盘 $D_R \subset\subset \Omega$，$f \in H(\Omega)$，$f(0) \neq 0$，$f(z) \neq 0, \forall z \in C_R = \partial D_R$

$z_1 \cdots z_n$ 表示 f 在 D_R 内的零点，则

$$\log|f(0)| = \sum_{k=1}^{N} \log\left(\frac{|z_k|}{R}\right) + \frac{1}{2\pi}\int_0^{2\pi} \log|f(Re^{i\theta})|\,d\theta$$

若 C_R 上有零点依然成立.

$g(z) = \dfrac{f(z)}{(z-z_1)\cdots(z-z_N)}$，$\log|f(0)| = \log|g(0)| + \sum_{k=1}^{N} \log|z_k|$

即证明，$\log|g(0)| = \frac{1}{2\pi}\int_0^{2\pi} \log|f(Re^{i\theta})|\,d\theta$

$$- \sum_{k=1}^{N} \log R$$

只用证明：对 $g(z)$ 在 D_R 无根及 $z-w$ $(|w|<R)$ 成立即可.

$\log|g(z)|$ 调和，即证：

$$\log|w| = \log\frac{|w|}{R} + \frac{1}{2\pi}\int_0^{2\pi} \log|Re^{i\theta} - w|\,d\theta$$

$\Leftrightarrow \quad 0 = \frac{1}{2\pi}\int_0^{2\pi} \log |1-\frac{w}{R}e^{i\theta}| d\theta$

$F(z) = 1-\frac{w}{R}z$ on $B(0,1)$ 即可.

(2) $n(r)$ 为 f 在 $B(0,r)$ 内零点数

$z_1 \cdots z_N$ 是 f 在 $B(0,R)$ 内的零点，则

$$\int_0^R \frac{n(r)}{r}dr = \sum_{k=1}^N \log|\frac{R}{z_k}|$$

$$\sum_{k=1}^N \log|\frac{R}{z_k}| = \sum_{k=1}^N \int_{|z_k|}^R \frac{dr}{r} = \int_0^R \sum_{k=1}^N \chi_{(z_k,R]}\frac{dr}{r} = \int_0^R \frac{n(r)}{r}dr$$

(3) $\rho_f = \inf\{\rho>0 \mid \exists A,B>0, \forall z\in\mathbb{C}, |f(z)|\le A\cdot e^{B|z|^\rho}\}$ 增长阶

$\rho_f \le \rho$，则

(a) $\exists\ C>0$, r充分大时有 $n(r)\le C r^\rho$

(b) $z_1, z_2\cdots$ 是 f 零点，$z_k\neq0$，$\forall\ s>\rho$，

$$\sum_{k=1}^\infty \frac{1}{|z_k|^s}<\infty$$

(a) 若 $f(0)=0$，考虑 $F(z)=f(z)/z^l$，l为 0 的阶数，

$n_f-n_F=\text{const}$，故不妨 $f(0)\neq0$

$$\int_0^R n(x)\cdot\frac{dx}{x} = \frac{1}{2\pi}\int_0^{2\pi}\log|f(Re^{i\theta})|d\theta - \log|f(0)|$$

$$\le \frac{1}{2\pi}\int_0^{2\pi}\log(Ae^{BR^\rho})d\theta - \log|f(0)| < C'\cdot r^\rho$$

$$\int_r^{2r} n(x)\frac{dx}{x} \ge n(r)\log 2 \Rightarrow n(r)\le C\cdot r^\rho$$

(b) $$\sum_{|z_k|\ge1}|z_k|^{-s} = \sum_{j=0}^\infty \sum_{2^j\le|z_k|\le 2^{j+1}} |z_k|^{-s}$$

$$\le \sum_{j=0}^\infty 2^{-js}\, n(2^{j+1})$$

$$\le \sum_{j=0}^\infty 2^{-js}\cdot C\cdot 2^{(j+1)\rho} \le C'\sum_{j=0}^\infty (2^{\rho-s})^j<\infty$$

(4)(a) $\sum|a_n|<\infty \Rightarrow \prod_{n=1}^\infty(1+a_n)$ 收敛

(b) $F_n \in H(\Omega)$, $\exists\, C_n > 0$, $\forall z \in \Omega$, $|F_n(z) - 1| \le C_n$, $\sum C_n < \infty$

(i) $\prod_{n=1}^{\infty} F_n(z) \rightrightarrows F(z) \in H(\Omega)$

(ii) $\forall n$, $F_n(z) \ne 0 \Rightarrow \frac{F'(z)}{F(z)} = \sum_{n=1}^{\infty} \frac{F_n'(z)}{F_n(z)}$

$G_N(z) = \prod_{n=1}^{N} F_n(z) \Rightarrow \frac{G_N'}{G_N} = \sum_{n=1}^{N} \frac{F_n'}{F_n}$

(5) (a) 典范因子

$$E_0(z) = 1 - z, \quad E_k(z) = (1-z)\, e^{z + \frac{z^2}{2} + \cdots + \frac{z^k}{k}}$$

则 $|z| \le 1/2 \Rightarrow |1 - E_k(z)| \le C \cdot |z|^{k+1}$,

$|z| \le 1/2$, $E_k(z) = e^{\log(1-z) + z + z^2/2 + \cdots + z^k/k} = e^{w}$

$\Rightarrow |w| \le 2|z|^{k+1} \le 1$, $|1 - E_k(z)| \le c'|w| \le c|z|^{k+1}$

(b) Theorem

$\forall$ 复数列 $\{a_n\}$, $|a_n| \to \infty$ as $n \to \infty$, 则

(i) $\exists$ 整函数 f, f 的所有零点为 $\{a_n\}$

(ii) 任意满足该条件的整函数形如 $f(z)e^{g(z)}$, g 为整函数.

(6) (a) $|z| \le 1/2 \Rightarrow |E_k(z)| \ge e^{-c|z|^{k+1}}$

(b) $\forall s$, $\rho_0 < s < k+1$, $\left|\prod_{n=1}^{\infty} E_k(z/a_n)\right| \ge e^{-c|z|^{s}}$

这里 $z \in \left(\bigcup_{n=1}^{\infty} B(a_n, |a_n|^{-k-1})\right)^c$

$$\prod_{n=1}^{\infty} E_k(z/a_n) = \prod_{|a_n| \le 2|z|} E_k(z/a_n) \cdot \prod_{|a_n| > 2|z|} E_k(z/a_n)$$

$$\prod_{|a_n| > 2|z|} |E_k(z/a_n)| \ge \prod_{|a_n| > 2|z|} e^{-c|z/a_n|^{k+1}}$$

$$\ge e^{-c'|z|^{k+1} \sum_{|a_n| > 2|z|} |z|^{-k-1}} \ge e^{-c''|z|^{s}}$$

$$\prod_{|a_n| \le 2|z|} |E_k(z/a_n)| \ge \prod_{|a_n| \le 2|z|} \left|1 - \frac{z}{a_n}\right| \prod_{|a_n| \le 2|z|} e^{-c'|z/a_n|^{k}}$$

$$\prod_{|a_n| \le 2|z|} e^{-c'|z/a_n|^{k}} \ge e^{-c'|z|^{k}} \cdot c'' \cdot |z|^{s-k} \sum |a_n|^{-s}$$

$$\geq e^{-C''|z|^s}$$

$$\prod_{|a_n|\leq 2|z|} \left|1-\frac{z}{a_n}\right| \geq \prod_{|a_n|\leq 2|z|} \left|\frac{a_n-z}{a_n}\right| \geq \prod_{|a_n|\leq 2|z|} |a_n|^{-k-2}$$

(c) $\exists\, r_n \in \mathbb{R}$, $r_n \to \infty$, s.t.

$$\left|\prod_{n=1}^{\infty} E_k(z/a_n)\right| \geq e^{-C|z|^s}, \quad |z| = r_m$$

(d) $g \in H(\mathbb{C})$，$u = \mathrm{Re}(g)$，对一列 $r_n \uparrow \infty$，$|z| = r_n$ 时，$u(z) \leq C r_n^s$

$\Rightarrow g$ 是不大于 s 阶的多项式.

$$g(z) = \sum_{n=0}^{\infty} a_n z^n$$

$$\frac{1}{2\pi}\int_0^{2\pi} g(re^{i\theta})\, e^{-in\theta}\, d\theta = \begin{cases} a_n r^n, & n \geq 0 \\ 0, & n < 0 \end{cases}$$

$$\Rightarrow \frac{1}{2\pi}\int_0^{2\pi} \overline{g(re^{i\theta})}\, e^{-in\theta}\, d\theta = 0, \quad n > 0$$

$$\Rightarrow a_n r^n = \frac{1}{\pi}\int_0^{2\pi} u(re^{i\theta})\, e^{-in\theta}\, d\theta$$

$$= \frac{1}{\pi}\int_0^{2\pi} (u(re^{i\theta}) - Cr^s)\, e^{-in\theta}\, d\theta$$

$$\Rightarrow |a_n| \leq \frac{1}{\pi r^n}\int_0^{2\pi} Cr^s - u(re^{i\theta})\, d\theta,$$

$\Rightarrow r \to \infty$, $a_n = 0$, for $n > s$

(e) Hardamard 定理.

f 是整函数，增长阶 ρ_0，$k \in \mathbb{N}$，$k \leq \rho_0 < k+1$，

$a_1, a_2, \cdots$ 表示 f 的非零零点，则

$$f(z) = e^{P(z)} \cdot z^m \cdot \prod_{n=1}^{\infty} E_k(z/a_n)$$

P 是次数 $\leq k$ 的多项式.

$E(z) = z^m \prod_{n=1}^{\infty} E_k(z/a_n)$，$n$ 充分大时

$|1-E_k(z/a_n)| \leq C \cdot \left|\frac{z}{a_n}\right|^{k+1}$，$\sum a_n^{-k-1}$ 收敛

$\Rightarrow f/E$ 全纯，无零点，$\Rightarrow \frac{f}{E} = e^{g(z)}$

$e^{\mathrm{Re}\, g} = |e^{g(z)}| = \left|\frac{f(z)}{E(z)}\right| \le c \cdot e^{c'|z|^s}$, $p_0 < s < k+1$

$\Rightarrow \mathrm{Re}\ g(z) \le C \cdot |z|^s$, $|z| = r_m$ $\Rightarrow$ g是不大于s阶多项式

6. Stein — Meromorphic function

(1) Casorati - Weierstrass

$f \in H(B(z_0, r) \setminus \{z_0\})$, has an essential singularity at z_0 $\Rightarrow$ $\overline{f(B(z_0,r)\setminus\{z_0\})} = \mathbb{C}$

(2) meromorphic on Ω

$f \in H(\Omega \setminus \{z_1, z_2, \cdots\})$,

$\{z_0, z_1, \cdots\}' = \phi$

$\lim\limits_{z \to z_i} f(z) = \infty$

(3) Argument principle

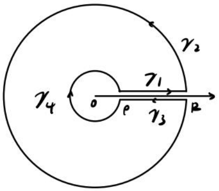
$\frac{1}{2\pi i} \int_\gamma \frac{f'}{f} dz$ = number of zeros of f inside γ − number of poles

(4) $f \in H(B(z_0, R))$, $f(z) = \sum\limits_{n=0}^{\infty} a_n (z - z_0)^n$, then

$$\frac{1}{2\pi r^n} \int_0^{2\pi} f(z_0 + re^{i\theta}) e^{-in\theta} d\theta \left(= \frac{1}{2\pi i} \int_\gamma \frac{f(\zeta)}{(\zeta - z_0)^{n+1}} d\zeta\right)$$

$$= \begin{cases} a_n, & n \ge 0 \\ 0, & n < 0 \end{cases}$$

1、 $I = \int_0^\infty \frac{x^{p-1}}{(1+x)^m} dx$, $0<p<m$, $m \in \mathbb{N}$



$$f(z) = \frac{z^{p-1}}{(1+z)^m}$$

$$f(x_{上}) = \frac{x^{p-1}}{(1+x)^m}$$

$$f(x_{下}) = f(x_{上}) \cdot \exp(i \cdot \Delta_c \mathrm{Arg}\,(p-1)\log z)$$

$$= \frac{x^{p-1}}{(1+x)^m} \cdot e^{2(p-1)\pi i}$$

$$= \frac{x^{p-1}}{(1+x)^m} \cdot e^{2p\pi i}$$

$\int_{\gamma_4} f(z)dz$, $\int_{\gamma_2} f(z)dz \to 0$

$$\Rightarrow \int_0^\infty f(x)dx = \frac{1}{1-e^{2p\pi i}} \cdot 2\pi i \,\mathrm{Res}\left(\frac{z^{p-1}}{(1+z)^m}, -1\right)$$

2. $I = \int_0^{2\pi} \frac{d\theta}{3+\cos\theta+2\sin\theta}$

$z = e^{i\theta}$, $I = \int \frac{\frac{1}{iz}dz}{3+\frac{z+1/z}{2}+\frac{z-1/z}{i}}$

$$= 2\int_{|z|=1} \frac{dz}{6iz + iz^2 + i + 2z^2 - 2}$$

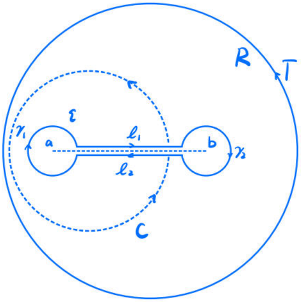
$$= 2\int_{|z|=1} \frac{dz}{(i+2)z^2 + 6iz + i - 2} = 4\pi i \cdot \frac{1}{4i} = \pi$$

3. $f \in \mathbb{C} \setminus \{a_1, \cdots, a_n\}$, $a_k \notin [a,b]$, $r, s \in (-1,1)$

$s \neq 0$, $r+s \in \mathbb{Z}$, 若 $\lim\limits_{z\to\infty} z^{r+s+1} f(z) = A \neq \infty$, 则

$$\int_a^b (x-a)^r (b-x)^s f(x)\,dx = -\frac{A\pi}{\sin s\pi} + \frac{\pi}{e^{-s\pi i}\sin s\pi}\sum_{k=1}^{n} Res(F, a_k)$$

$$F(z) = (z-a)^r (b-z)^s f(z)$$



1、F在该区域有单值分支

$$\Delta_C F = f(z)\,\Delta_C \exp(r\log(z-a) + s\log(b-z))$$

$$= f(z)\cdot|z-a|^r\cdot|b-z|^s \cdot\left(\exp[i\Delta_C(r\,Arg(z-a)+s\,Arg(b-z))]-1\right)$$

$$\Delta_C Arg(z-a) = \Delta_C Arg(b-z) = 2k\pi$$

$$\Rightarrow \Delta_C F = 0$$

2. $F(x_{下}) = \Delta_C F(x) + F(x_{上})$

$$= f(x)\cdot|z-a|^r\cdot|b-z|^s\cdot\exp\left(i\Delta_C(r\,Arg(z-a)+s\,Arg(b-z))\right)$$

$$= F(x_{上})\cdot\exp(2\pi r i)$$

$$\int_{l_2} F\,dz = -\int_{a+\varepsilon}^{b-\varepsilon} F(x_{下})\,dx = -e^{2\pi r i}\int_{l_1} F\,dz$$

$$= -e^{-2\pi s i}\int_{l_1} F\,dz$$

$$\lim_{z\to a}(z-a)F(z) = 0 \Rightarrow \int_{\gamma_1} F\,dz \to 0 \quad as\ \varepsilon\to 0$$

$$\lim_{z\to b}(b-z)F(z) = 0 \Rightarrow \int_{\gamma_2} F\,dz \to 0 \quad as\ \varepsilon\to 0$$

$$\lim_{z\to\infty} z\cdot F(z) = \lim_{z\to\infty} z(z-a)^r(b-z)^s f(z)$$

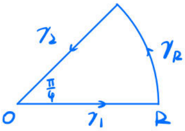
$$= e^{-s\pi i}\lim_{z\to\infty} z^{r+s+1} f(z) = e^{-s\pi i}A$$

$$\Rightarrow \int_T F\,dz \to 2\pi i\, e^{-s\pi i}A$$

$$\Rightarrow \int_a^b F(x)\,dx = \frac{1}{1-e^{-2\pi s i}}\left(2\pi i\sum Res(F, a_k) - 2\pi i\, e^{-s\pi i}A\right)$$

4. $\int_0^\infty \cos x^2 \, dx$, $\int_0^\infty \sin x^2 \, dx$

$f(z) = e^{iz^2}$



$$\int_{\gamma_R} f(z)\,dz$$

$$= \int_0^{\frac{\pi}{4}} i \cdot e^{iR^2 e^{2i\theta}} \cdot R e^{i\theta} \, d\theta$$

$$= \int_0^{\frac{\pi}{4}} i e^{i\theta} . R \; e^{iR^2 \cos 2\theta - R^2 \sin 2\theta} \, d\theta$$

$$\left| \int_{\gamma_R} f(z)\,dz \right| \le \int_0^{\frac{\pi}{4}} R \cdot e^{-R^2 \sin 2\theta} \, d\theta \le R \int_0^{\frac{\pi}{4}} e^{-R^2 \frac{4}{\pi}\theta} \, d\theta$$

$$= \frac{\pi}{4R} \left(1 - e^{-R^2} \right) \to 0$$

$$\int_{\gamma_2} f(z)\,dz = \int_R^0 e^{ir^2 e^{\frac{\pi}{2}i}} \cdot e^{\frac{\pi}{4}i} \, dr = -e^{\frac{\pi}{4}i} \int_0^R e^{-r^2} \, dr$$

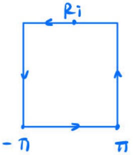
$$\to -\frac{\sqrt{\pi}}{2} e^{i\pi/4}$$

故 $\int_0^\infty \cos x^2 \, dx + i \int_0^\infty \sin x^2 \, dx = \frac{\sqrt{\pi}}{2} e^{i\pi/4}$

$$\Rightarrow \int_0^\infty \cos x^2 \, dx = \int_0^\infty \sin x^2 \, dx = \frac{\sqrt{2\pi}}{4}$$

5. $I = \int_0^\pi \frac{x \sin x}{1 - 2a\cos x + a^2} \, dx$, $a > 1$

$f(z) = \frac{z}{a - e^{-iz}}$



$$\int_{-\pi}^{\pi} \frac{x}{a - \cos x + i \sin x} \, dx$$

$$+ i \cdot \int_0^R \frac{\pi + iy}{a + e^y} \, dy - i \cdot \int_0^R \frac{iy - \pi}{a + e^y} \, dy$$

$$- \int_{-\pi}^{\pi} \frac{x + Ri}{a - e^R e^{-ix}} \, dx = 2\pi i \cdot \frac{\log a}{a}$$

$$\int_{-\pi}^{\pi} \frac{x}{a-\cos x+i\sin x}\,dx = \int_{-\pi}^{\pi} \frac{x(a-\cos x)-ix\sin x}{1-2a\cos x+a^2}$$

$$\int_{0}^{R} \frac{\pi+iy}{a+e^y}\,dy - \int_{0}^{R} \frac{iy-\pi}{a+e^y}\,dy$$

$$= \int_{0}^{R} \frac{2\pi}{a+e^y}\,dy \to 2\pi\cdot\frac{1}{a}\log(a+1)$$

$$\int_{-\pi}^{\pi} \frac{x+Ri}{a-e^R\cdot e^{-ix}}\,dx \to 0$$

$$\Rightarrow I = -\frac{1}{2}\,\mathrm{Im}\left(2\pi i\cdot\frac{\log a}{a} - 2\pi i\cdot\frac{1}{a}\log(a+1)\right)$$

$$= \frac{\pi}{a}\log\left(1+\frac{1}{a}\right)$$

6. $I = \int_{0}^{\infty} \frac{\log(1+x^2)}{1+x^2}\,dx$

$$= \int_{0}^{\infty} \log(1+x^2)\,d\arctan x$$

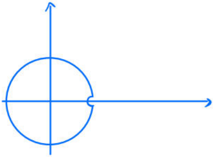
$$= \int_{0}^{\frac{\pi}{2}} \log\left(\frac{1}{\cos^2\theta}\right)d\theta$$

$$= 2\int_{0}^{\frac{\pi}{2}} \log\sin\theta\,d\theta$$

7. $\int_{0}^{\frac{\pi}{2}} \log\sin\theta\,d\theta$

$$\int_0^{\frac{\pi}{2}} \log|e^{i\theta}-1|\,d\theta = Re \int_0^{\frac{\pi}{2}} \log(e^{i\theta}-1)\,d\theta$$

$$\int_0^{\frac{\pi}{2}} \log(e^{i\theta}-1)\,d\theta = \int_\gamma \log(z-1)\,\frac{1}{iz}\,dz$$



$$f(z) = \frac{\log(z-1)}{z}$$

$$\int_{|z|=1} f(z)\,dz = 2\pi i \log(-1)$$

$$= -2\pi^2$$

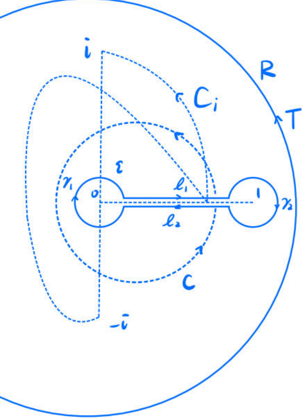
$$\int_{|z|=1} f(z)\,dz = i\int_0^{2\pi} \log(e^{i\theta}-1)\,d\theta$$

$$\int_0^{2\pi} \log|e^{i\theta}-1| = Im\left(\int_{|z|=1} f(z)\,dz\right) = 0$$

$$\Rightarrow \int_0^{\pi} \log|e^{i\theta}-1|\,d\theta = 0 \quad \Rightarrow \int_0^{\pi} \log\left(2\sin\frac{\theta}{2}\right)d\theta = 0$$

$$\Rightarrow \int_0^{\frac{\pi}{2}} \log\sin\theta\,d\theta = -\frac{\pi}{2}\log 2$$

8. $I = \int_0^1 \frac{x^{1-p}(1-x)^p}{1+x^2}\,dx \quad , \quad -1 < p < 2$



$$f(z) = \frac{z^{1-p}(1-z)^p}{1+z^2}$$

$$f(x_{\text{下}}) = f(x_{\text{上}})\, e^{i((1-p)\Delta_C \operatorname{Arg} z + p\Delta_C \operatorname{Arg}(1-z))}$$

$$= \frac{x^{1-p}(1-x)^p}{1+x^2} \cdot e^{-2\pi p i}$$

$$\int_{l_2} f(z) = -e^{-2\pi p i} \int_{l_1} f(z)$$

$z \cdot f(z) \to 0$ as $z \to 0$

$(1-z) \cdot f(z) \to 0$ as $z \to 1$

故 $\int_{\gamma_1} f(z) \to 0$, $\int_{\gamma_2} f(z) \to 0$

$$\lim_{z\to\infty} z \cdot f(z) = \lim_{z\to\infty} z^{-p}(1-z)^p$$

$$= \lim_{z\to\infty} \exp(-p\log|z| + p\log|1-z| - i\,p\operatorname{Arg} z + i\,p\operatorname{Arg}(z-1) - i\,p\pi\,)$$

$$= e^{-ip\pi}$$

$$\int_T f(z)\,dz \to 2\pi i\; e^{ip\pi}$$

$$\operatorname{Res}(f(z), i) = \frac{i^{1-p}(1-i)^p}{2i}$$

$$= \frac{2^{\frac{p}{2}}}{2i} \exp((1-p)\operatorname{Arg} z + p\operatorname{Arg}(1-z))$$

$$= \frac{2^{p/2}}{2i} \exp((1-p)\cdot\frac{\pi}{2} - p\cdot\frac{1}{4}\pi) = \frac{2^{\frac{p}{2}}}{2} e^{-\frac{3}{4}p\pi i}$$

$$\operatorname{Res}(f(z), -i) = \frac{(-i)^{1-p}(1+i)^p}{-2i}$$

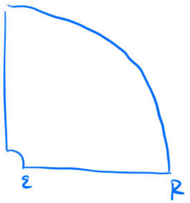
$$= -\frac{2^{\frac{p}{2}}}{2i} \exp(i[(1-p)\operatorname{Arg} z + p\operatorname{Arg}(1-z)])$$

$$= -\frac{2^{p/2}}{2i} \exp\left(i\left((1-p)\frac{3}{2}\pi + p\frac{\pi}{4}\right)\right) = \frac{2^{p/2}}{2} \exp\left(-\frac{5}{4}p\pi i\right)$$

故 $I = \frac{1}{1-e^{-2\pi p i}} \left(\pi i \cdot 2^{\frac{p}{2}}\left(e^{-\frac{3}{4}p\pi i} + e^{-\frac{5}{4}p\pi i}\right) - 2\pi i\, e^{-ip\pi}\right)$

$$= \frac{2\pi i}{e^{\pi p i} - e^{-\pi p i}} \left(2^{\frac{p}{2}} \cos\frac{p}{4}\pi - 1\right) = \frac{\pi}{\sin p\pi} \left(2^{\frac{p}{2}} \cos\frac{p\pi}{4} - 1\right)$$

9. $I = \int_0^\infty \frac{\log x}{x^2-1}\, dx$



$$\Rightarrow \int_0^\infty \frac{\log x}{x^2-1} - i\int_0^\infty \frac{\log iy}{-y^2-1}\, dy = 0$$

$$\Rightarrow \int_0^\infty \frac{\log x}{x^2-1} - \frac{\pi}{2}\int_0^\infty \frac{1}{y^2+1} dy + i\int_0^\infty \frac{\log y}{y^2+1} = 0$$

$$\Rightarrow I = \frac{\pi^2}{4}, \quad \int_0^\infty \frac{\log y}{y^2+1} = 0$$

10. U, V 共形等价, U SC $\Rightarrow$ V SC

但 U SC + f Holomorphic $\not\Rightarrow$ $f(U)$ SC

$f: U \to f(U)$ 共形, 则

$$\mathrm{Area}(f(U)) = \iint_U |f'(z)|^2 \, dx\,dy$$

11. $f \in H(\mathbb{C})$，无零点，则 $\exists\, g \in H(\mathbb{C})$，$f = e^{g(z)}$

$$\frac{f'}{f} \in H(\mathbb{C}) \quad , \quad g(z) = \int_{z_0}^{z} \frac{f'(\zeta)}{f(\zeta)} d\zeta + c_0$$

$$e^{c_0} = f(z_0) \quad \Rightarrow \quad g'(z) = \frac{f'(z)}{f(z)}$$

$$\Rightarrow (f e^{-g})' = 0 \Rightarrow f = c \cdot e^{g} \Rightarrow f = e^{g}$$